MATH2401 §1 Functions of More than One Independent Variable

Partial Differentiation

Definition: Let $z = f(x, y)$ be a function of two independent variables x and y. Then z has derivatives with respect to both x and y. These are the *partial derivatives*

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}.$$

To determine these derivatives, since x and y vary independently, we treat y as a constant to evaluate $\partial z/\partial x$ and treat x as a constant to evaluate $\partial z/\partial y$.

Important: Can also use suffix notation, to save on effort,

$$z_x \text{ for } \frac{\partial z}{\partial x}, \qquad z_y \text{ for } \frac{\partial z}{\partial y}.$$

Example: Find the partial derivatives of $z = 3x^2 \sin y + x^3$.

$$\Big[\text{Answer:} \quad z_x = 6x \sin y + 3x^2, \quad z_y = 3x^2 \cos y, \Big]$$

Chain Rule for Partial Differentiation

Theorem: Let $z = f(u, v)$ where $u = u(x, y)$ and $v = v(x, y)$ are functions of the independent variables x and y. Then z is also a function of x and y and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Proof: Let δx be a small change in x, *while keeping y constant.* Let δz, δu and δv represent the corresponding changes in z, u, v respectively.

$$\begin{aligned} \delta z &= f(u + \delta u, v + \delta v) - f(u, v) \\ &= f(u + \delta u, v + \delta v) - f(u, v + \delta v) + f(u, v + \delta v) - f(u, v) \end{aligned}$$

$$\frac{\delta z}{\delta x} = \frac{f(u + \delta u, v + \delta v) - f(u, v + \delta v)}{\delta u}.\frac{\delta u}{\delta x} + \frac{f(u, v + \delta v) - f(u, v)}{\delta v}.\frac{\delta v}{\delta x}$$

Taking the limit $\delta x \to 0$ gives

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}.$$

There is a similar proof for $\partial z/\partial y$.

Note also the other examples of the chain rule covered in lectures.

Gradient of a Function

Definition: Consider $\phi = \phi(x, y, z)$ with x, y, z independent. The gradient of ϕ, denoted $\nabla\phi$ or grad ϕ is given by

$$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{\mathbf{n}}.$$

The vector $\hat{\mathbf{n}}$ is the unit normal to the surface defined by $\phi(x, y, z) = c$ constant, in the direction of increasing ϕ, and $\partial/\partial n$ is the directional derivative in this direction.

Theorem: If (x, y, z) define a Cartesian coordinate system spanned by orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

Consider two neighbouring points P, Q (see figure).

- Let $\hat{\mathbf{s}}$ be the unit vector in the direction PQ.
- Let s measure distance along PQ.
- Let $\hat{\mathbf{n}}$ be the unit normal to the surface $\phi = \phi(P) = \phi(x(P), y(P), z(P))$ at P.
- Let n measure distance in the direction of $\hat{\mathbf{n}}$.
- Let N be the point where a line in the direction of $\hat{\mathbf{n}}$ intersects the surface $\phi = \phi(Q)$.

Definition: $\partial\phi/\partial s$ is the *directional derivative* of the function ϕ at P in the direction $\hat{\mathbf{s}}$.

$$\frac{\partial\phi}{\partial s} = \lim_{Q\to P}\frac{\phi(Q)-\phi(P)}{|PQ|} = \lim_{Q\to P}\frac{\phi(N)-\phi(P)}{|PN|}\cdot\left|\frac{PN}{PQ}\right| = \frac{\partial\phi}{\partial n}\cos\theta.$$

The maximum value of the directional derivative at P is therefore in the direction $\hat{\mathbf{n}}$. Also

$$\frac{\partial\phi}{\partial s} = \hat{\mathbf{s}}\cdot\hat{\mathbf{n}}\,\frac{\partial\phi}{\partial n} = \hat{\mathbf{s}}\cdot\nabla\phi \qquad \text{Directional derivative equation.}$$

Letting $\nabla\phi = \lambda\mathbf{i}+\mu\mathbf{j}+\nu\mathbf{k}$, then from the dir. deriv. eqn. $\lambda = \mathbf{i}\cdot\nabla\phi = \dfrac{\partial\phi}{\partial x}$, $\mu = \mathbf{j}\cdot\nabla\phi = \dfrac{\partial\phi}{\partial y}$ etc.

$$\text{Hence} \qquad \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$

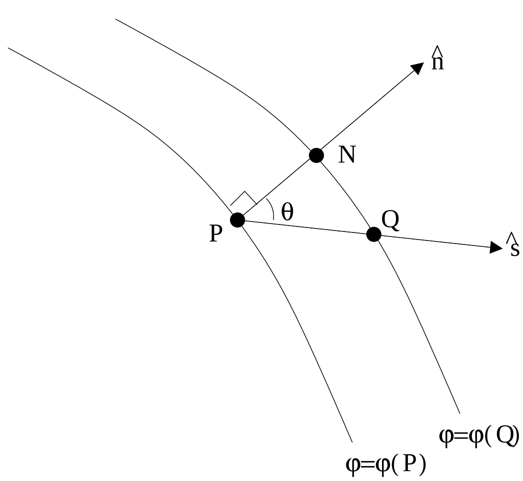


Figure: Illustrating the geometric construction of $\nabla\phi$ at P.

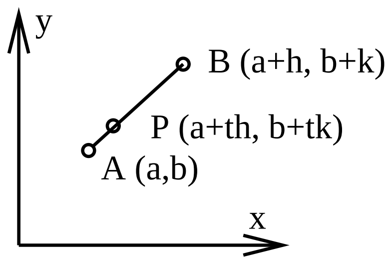


Figure: Illustrating the construction of the one-dimensional function used in the proof below.

Taylor Series for Functions of Two Variables

Theorem: Let $f(x, y)$ be a function of two independent variables. Provided all the required partial derivatives exist throughout the region of interest, then $f(x, y)$ can be expanded around the point (a, b) in a Taylor series as follows

$$f(a+h, b+k) = f(a, b) + [(\zeta.\nabla)f](a, b) + \frac{1}{2!}\left[(\zeta.\nabla)^2 f\right](a, b) + ... + \frac{1}{n!}\left[(\zeta.\nabla)^n f\right](a, b) + ...$$

where $\zeta = (h, k)$. Note that the operator $(\zeta.\nabla)$ may be expanded as follows

$$\begin{aligned}
(\zeta.\nabla) &= h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \\
(\zeta.\nabla)^2 &= h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x \partial y} + k^2\frac{\partial^2}{\partial y^2} \\
(\zeta.\nabla)^n &= h^n\frac{\partial^n}{\partial x^n} + ... + \begin{pmatrix} n \\ i \end{pmatrix} h^{(n-i)}k^i\frac{\partial^n}{\partial x^{(n-i)}\partial y^i} + ... + k^n\frac{\partial^n}{\partial y^n}
\end{aligned}$$

(The same equation applies for **three or more variables** with suitably redefined ζ and ∇.)

Proof: Consider the function $F(t) = f(a + th, b + tk)$, $t \in [0, 1]$, constructed as in the diagram above. Note that $F(0) = f(a, b)$ and $F(1) = f(a + h, b + k)$. As it is a function of a single variable t, we can expand $F(t)$ in a Taylor series,

$$F(t) = F(0) + tF'(0) + \frac{1}{2!}t^2F''(0) + ... + \frac{1}{n!}t^nF^{(n)}(0) + ... \qquad (*)$$

Using the chain rule

$$F'(t) = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = (\zeta.\nabla)f, \qquad F''(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)F'(t) = (\zeta.\nabla)^2 f.$$

By induction it can be shown that $F^{(n)}(t) = (\zeta.\nabla)^n f$. Evaluating these expressions at $t = 0$ is equivalent to evaluating at (a, b). Evaluating (*) at $t = 1$ and substituting for F get

$$f(a+h, b+k) = f(a, b) + [(\zeta.\nabla)f](a, b) + \frac{1}{2!}\left[(\zeta.\nabla)^2 f\right](a, b) + ... + \frac{1}{n!}\left[(\zeta.\nabla)^n f\right](a, b) + ...$$

Example: Expand $f(x, y) = x^2 + 3y - 2$ in powers of $x - 1$, $y + 2$.

$$\left[\text{Answer: } f(x, y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)\right]$$

Mean Value Theorem

For functions of two (or more) variables there is also a mean value theorem.

Theorem: If $f(x,y)$ is continuous in a closed region R of the (x,y) plane, including the straight line joining the points (a,b) and $(a+h,b+k)$, and the partial derivatives f_x and f_y exist in the open region of R, then there exists $\theta \in (0,1)$ such that

$$f(a+h,b+k) - f(a,b) = hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k).$$

Proof: Consider the function $F(t) = f(a+th, b+tk)$. The mean value theorem for a function of one variable implies that there exists $\theta \in (0,1)$ such that

$$F(1) - F(0) = F'(\theta).$$

and using the chain rule

$$F'(\theta) = \left[f_x \frac{dx}{dt} + f_y \frac{dy}{dt}\right]_{t=\theta} = hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k).$$

Since $F(1) = f(a+h,b+k)$ and $F(0) = f(a,b)$, the result is proved.

Taylor's Theorem

Similarly, the following version of Taylor's theorem (truncated Taylor series) holds:
Theorem: Assuming all the required derivates exist, and that $f(x,y)$ is defined on a suitable region R as above, then there exists $\theta \in (0,1)$ such that

$$f(a+h,b+k) = f(a,b) + \left[(\zeta.\nabla)f\right](a,b) + ... + \frac{1}{n!}\left[(\zeta.\nabla)^n f\right](a,b) + \frac{1}{(n+1)!}\left[(\zeta.\nabla)^{(n+1)} f\right](a+\theta h, b+\theta k),$$

where $\zeta = h\mathbf{i} + k\mathbf{j}$.

Proof: This can be proved by a straighforward adaptation of the Taylor's series proof above, again using $F(t) = f(a+th, b+tk)$, using the single variable Taylor's theorem: There exists $\theta \in (0,1)$ such that

$$F(1) = F(0) + F'(0) + \frac{1}{2!}F''(0) + .. + \frac{1}{n!}F^{(n)}(0) + \frac{1}{(n+1)!}F^{(n+1)}(\theta).$$

Critical Points: Definitions

Critical Point: A point (x_0, y_0) is a critical point of a function $z = f(x, y)$ if

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Relative Maximum: A function $z = f(x, y)$ has a relative maximum at (x_0, y_0) if $f(x, y) < f(x_0, y_0)$ for all points sufficiently close to (x_0, y_0).

Relative Minimum: As above but $f(x, y) > f(x_0, y_0)$ for nearby points.

Saddle Point: A saddle point is a critical point (x_0, y_0) for which there exists θ_1, θ_2 such that such that $f(x + \delta\cos\theta_1, y + \delta\sin\theta_1) > f(x_0, y_0)$ and $f(x + \delta\cos\theta_2, y + \delta\sin\theta_2) < f(x_0, y_0)$ for arbitrarily small δ.

Properties of Critical Points

Definition: The quantity

$$\Delta(x, y) = f_{xx} f_{yy} - f_{xy}^2,$$

is known as the *discriminant* of the function $z = f(x, y)$.

Theorem:

1. A necessary condition for a relative maximum or minimum at (x_0, y_0) is that (x_0, y_0) is a critical point.
2. If both $\Delta > 0$ and $f_{xx} < 0$ at a critical point (x_0, y_0) then it is a relative maximum.
3. If instead $\Delta > 0$ and $f_{xx} > 0$ at a critical point (x_0, y_0) it is a relative minimum.
4. If $\Delta < 0$ at a critical point (x_0, y_0) it is a saddle point.

Example: Find and classify the critical points of

$$f(x, y) = \frac{1}{3}(x^3 + y^3) - x^2 - y^2.$$

Answer:	Relative maximum at	$(0, 0)$
	Saddle point at	$(0, 2)$
	Saddle point at	$(2, 0)$
	Relative minimum at	$(2, 2)$

Proof of (1): At a maximum or minimum, the tangent plane to $z = f(x, y)$ at (x_0, y_0) is parallel to the (x, y) plane. Therefore along any direction $\hat{\mathbf{s}} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$ in the tangent plane

$$\frac{\partial f}{\partial s} = (\hat{\mathbf{s}}.\nabla)f = \frac{\partial f}{\partial x}\cos\phi + \frac{\partial f}{\partial y}\sin\phi = 0.$$

This is true for all possible values of ϕ, so at (x_0, y_0), $\partial f/\partial x = \partial f/\partial y = 0$, and hence it is a critical point.

Proof of (2): Let $(x_0 + h, y_0 + k)$ be a point in R. Taylor's theorem gives

$$\begin{aligned} f(x_0+h, y_0+k) - f(x_0, y_0) &= hf_x(x_0, y_0) + kf_y(x_0, y_0) + \frac{1}{2}\left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]\Big|_{(x_0+th, y_0+tk)} \\ &= \frac{1}{2}f_{xx}\left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + k^2\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right]\Bigg|_{(x_0+th, y_0+tk)} \\ &= \frac{1}{2}f_{xx}\left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + k^2\frac{\Delta}{f_{xx}^2}\right]\Bigg|_{(x_0+th, y_0+tk)} \end{aligned}$$

Note that the final term on the RHS is evaluated at $(x_0 + th, y_0 + tk)$, with $0 < t < 1$.
If $\Delta > 0$ and $f_{xx} < 0$ at (x_0, y_0) then because the function and its derivatives are continuous, the RHS above must be negative for all (sufficiently small) choices of h and k. Therefore

$$f(x_0 + h, y_0 + k) < f(x_0, y_0) \quad \text{for all small enough } h,\ k$$

and (x_0, y_0) is a *relative maximum.*

Proof of (3): Easy modification of the above.

Proof of (4): Three cases need to be considered:

Case 1: $f_{xx} \neq 0$

In this case we have

$$\begin{aligned}
f(x_0+h, y_0+k) - f(x_0, y_0) &= h f_x(x_0, y_0) + k f_y(x_0, y_0) + \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right]\Big|_{(x_0+th, y_0+tk)} \\
&= \frac{1}{2} f_{xx} \left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + k^2 \frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}^2}\right]\Bigg|_{(x_0+th, y_0+tk)} \\
&= \frac{1}{2} f_{xx} \left[\left(h + k\frac{f_{xy}}{f_{xx}}\right)^2 + k^2 \frac{\Delta}{f_{xx}^2}\right]\Bigg|_{(x_0+th, y_0+tk)}
\end{aligned}$$

Writing $h = \delta\cos\theta$, $k = \delta\sin\theta$ gives

$$f(x_0+\delta\cos\theta, y_0+\delta\sin\theta) - f(x_0, y_0) = \frac{\delta^2}{2} f_{xx}\left[\left(\cos\theta + \sin\theta\frac{f_{xy}}{f_{xx}}\right)^2 + \sin^2\theta\frac{\Delta}{f_{xx}^2}\right]\Bigg|_{(x_0+t\delta\cos\theta, y_0+t\delta\sin\theta)}.$$

If we choose $\theta_1 = -\tan^{-1}(f_{xx}/f_{yy})$ and $\theta_2 = 0$ then we have (taking e.g. $f_{xx} > 0$ and taking δ small)

$$\begin{aligned}
& f(x_0+\delta\cos\theta_1, y_0+\delta\sin\theta_1) - f(x_0, y_0) < 0 \\
\text{and} \quad & f(x_0+\delta\cos\theta_2, y_0+\delta\sin\theta_2) - f(x_0, y_0) > 0
\end{aligned}$$

and we have thus shown that (x_0, y_0) is a saddle point.

Case 2: $f_{xx} = 0$, $f_{yy} \neq 0$

A minor variation of the above (extracting f_{yy} instead of f_{xx}) gives

$$\begin{aligned}
f(x_0+h, y_0+k) - f(x_0, y_0) &= h f_x(x_0, y_0) + k f_y(x_0, y_0) + \frac{1}{2}\left[2hk f_{xy} + k^2 f_{yy}\right]\Big|_{(x_0+th, y_0+tk)} \\
&= \frac{1}{2} f_{yy} \left[\left(k + h\frac{f_{xy}}{f_{yy}}\right)^2 + h^2 \frac{\Delta}{f_{yy}^2}\right]\Bigg|_{(x_0+th, y_0+tk)}
\end{aligned}$$

Inserting $h = \delta\cos\theta$ and $k = \delta\sin\theta$ and choosing $\theta_1 = -\tan^{-1}(f_{xy}/f_{yy})$ and $\theta_2 = 0$ then we have (taking e.g. $f_{yy} > 0$ and taking δ small)

$$\begin{aligned}
& f(x_0+\delta\cos\theta_1, y_0+\delta\sin\theta_1) - f(x_0, y_0) < 0 \\
\text{and} \quad & f(x_0+\delta\cos\theta_2, y_0+\delta\sin\theta_2) - f(x_0, y_0) > 0
\end{aligned}$$

and (x_0, y_0) is a saddle point in this case as well.

Case 3: $f_{xx} = 0$, $f_{yy} = 0$

In this case

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{1}{2}\left[2hk f_{xy}\right]\Big|_{(x_0+th, y_0+tk)}$$

Writing $h = \delta\cos\theta$ and $k = \delta\sin\theta$, and choosing $\theta_1 = \pi/4$, $\theta_2 = 3\pi/4$ causes the product hk to take different signs, so that for e.g. $f_{xy} > 0$ (and δ small)

$$\begin{aligned}
& f(x_0+\delta\cos\theta_1, y_0+\delta\sin\theta_1) - f(x_0, y_0) > 0 \\
\text{and} \quad & f(x_0+\delta\cos\theta_2, y_0+\delta\sin\theta_2) - f(x_0, y_0) < 0
\end{aligned}$$

in this case, hence (x_0, y_0) is a saddle point.

Lagrange Multipliers: Finding extreme values under constraints

Problem: Find an extreme value (maximum or minimum) of the function $f(x, y)$ under the *constraint* $g(x, y) = 0$.

Theorem: A *necessary* condition for a point (x_0, y_0) to achieve an extreme value of $f(x, y)$ subject to $g(x, y) = 0$ is that (x_0, y_0) is a critical point of the function

$$h(x, y) = f(x, y) + \lambda g(x, y),$$

where λ is an initially unknown constant that may be determined from the constraint condition $g(x, y) = 0$.

Definition: The constant λ is known a *Lagrange Multiplier*.

Proof:

The constraint $g(x, y) = 0$ defines a curve in the (x, y) plane. Suppose we parameterise this curve

$$x = X(t), \quad y = Y(t), \quad \text{so that} \quad g(X(t), Y(t)) = G(t) = 0.$$

Differentiating with respect to t using the chain rule,

$$\frac{dG}{dt} = 0, \quad \text{or} \quad g_x \frac{dX}{dt} + g_y \frac{dY}{dt} = 0, \qquad (*)$$

at all points along the curve $g(x, y) = 0$. For points along this same curve we can write

$$f(x, y) \;=\; f(X(t), Y(t)) \;=\; F(t).$$

For a maximum or minimum of $f(x, y)$ subject to $g(x, y) = 0$ we need to find a point where

$$F'(t) = 0 \quad \text{or} \quad f_x \frac{dX}{dt} + f_y \frac{dY}{dt} = 0. \qquad (**)$$

The conditions (*) and (**) can be rearranged to give

$$\frac{f_x}{f_y} = \frac{g_x}{g_y} = -\frac{dY/dt}{dX/dt},$$

hence at an extreme value of f subject to $g = 0$,

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} = -\lambda, \qquad \text{(say)}.$$

Therefore for a point (x_0, y_0) to give an extreme value of $f(x, y)$ subject to $g(x, y) = 0$, there exists λ such that

$$f_x + \lambda g_x = 0, \qquad \text{and} \qquad f_y + \lambda g_y = 0. \qquad \text{at } (x_0, y_0).$$

The point (x_0, y_0) is therefore a critical point of the function h defined above. A third equation, necessary if λ is to be determined, is provided by the constraint itself $g(x_0, y_0) = 0$.

Technique:

1. Find the location the critical points of $h(x, y)$ in terms of the unknown constant λ.
2. For each critical point in turn, use the constraint $g(x, y) = 0$ to eliminate λ.
3. Examine each critical point to determine whether it represents a maximum or minimum of $f(x, y)$ subject to $g(x, y) = 0$.

Example 1: Find the shortest distance from the origin to the straight line $ax + by + c = 0$.

$$\left[\text{Answer:} \quad \frac{|c|}{\sqrt{a^2 + b^2}}.\right]$$

Example 2: An open vessel has the form of a circular cylinder. Find the maximum volume V for a given surface area $S = S_0$.

$$\left[\text{Answer:} \quad V = \frac{S_0^{3/2}}{3\sqrt{3\pi}}\right]$$

Example 3: Find the minimum distance to the origin along the hyperbola $x^2 + 8xy + 7y^2 = 225$.

[Answer: 5]

Notes:

1. Not every critical point of $h(x, y)$ necessarily corresponds to a maximum or minimum of $f(x, y)$ subject to $g(x, y) = 0$.
2. The technique may be extended to functions of three or more independent variables. For example, the extreme values of $f(x, y, z)$ subject to $g(x, y, z) = 0$, if they exist, will occur at critical points of
$$h(x, y, z) = f(x, y, z) + \lambda g(x, y, z).$$
3. For functions of N variables we can have up to $N - 1$ constraints. For example, the extreme values of $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $k(x, y, z) = 0$ will occur at critical points of
$$h(x, y, z) = f(x, y, z) + \lambda g(x, y, z) + \mu k(x, y, z).$$
where the two multipliers λ, μ may be determined from the two constraints $g(x, y, z) = 0$ and $k(x, y, z) = 0$.

Example 4: An open rectangular trough has volume 32m^3. For what dimensions $x \times y \times z$ does it have minimum surface area?

[Answer: $4 \times 4 \times 2$ m.]

Example 5: Find the point on the plane $x - 2y + 3z = 5$ that lies closest to the origin.

$$\left[\text{Answer:} \quad \frac{5}{14}(1, -2, 3)\right]$$

MATH2401 §2 Calculus of Variations

What is the Calculus of Variations?

Many problems involve finding a function that maximizes or minimizes an integral expression.

One example is finding the curve giving the shortest distance between two points - a straight line, of course, in Cartesian geometry (but can you prove it?) but less obvious if the two points lie on a curved surface (the problem of finding *geodesics*.)

The mathematical techniques developed to solve this type of problem are collectively known as the *calculus of variations*.

Functionals and Extremals

Consider a definite integral that depends on an unknown function $y(x)$, as well as its derivative $y'(x) = dy/dx$,

$$I = \int_a^b F(x, y, y')\ dx.$$

A typical problem in the calculus of variations involve finding a particular function $y(x) = f(x)$ to maximize or minimize the integral I subject to boundary conditions $y(a) = A$ and $y(b) = B$.

The integral I is an example of a *functional*, which (more generally) is a mapping from a vector space of allowable functions to the reals.

The function $f(x)$ that maximizes or minimizes I while satisfying the boundary conditions $f(a) = A$, $f(b) = B$ is known as an *extremal function*.

Example Problem

For example, the curve $y(x) = f(x)$ that gives the shortest distance between the points $(0, 0)$ and $(1, 1)$ must minimize the functional

$$I = \int_0^1 ds = \int_0^1 (1 + (y')^2)^{1/2}\, dx \quad \text{subject to boundary conditions } \ y(0) = 0, \quad y(1) = 1.$$

The curve $f(x)$ is therefore an extremal of the functional I.

Obviously, $f(x) = x$, the straight line joining $(0, 0)$ and $(1, 1)$, but how do we prove this?

The Euler-Lagrange Equation, or Euler's Equation

A necessary condition for $y(x) = f(x)$ to be an extremal of the functional

$$I = \int_a^b F(x, y, y')\,dx, \quad \text{subject to} \quad y(a) = A,\ y(b) = B,$$

is that $y(x) = f(x)$ satsifies the second order ordinary differential equation defined by

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \quad \text{with b.c.s}\ \ y(a) = A,\ y(b) = B.$$

This is the *Euler-Lagrange equation*, or sometimes just Euler's equation.
Proof: Consider a set of functions close to $f(x)$, e.g. those given by

$$y(x) = f(x) + \epsilon\eta(x),$$

where $\eta(x)$ is any twice differentiable function of x with $\eta(a) = \eta(b) = 0$ so that $y(a) = A$ and $y(b) = B$, i.e. y still satisfies the boundary conditions. Here ϵ is a small parameter.

Definition: The function $\epsilon\eta(x)$ is known as *the variation in* $f(x)$.

If $y(x) = f(x) + \epsilon\eta(x)$ is inserted into the functional I, the result may be considered as a function of ϵ,

$$I(\epsilon) = \int_a^b F(x, f + \epsilon\eta, f' + \epsilon\eta')\,dx.$$

Because $f(x)$ is an extremal of I, I is minimized or maximized when $\epsilon = 0$, therefore

$$I'(\epsilon) = 0, \qquad \text{when} \qquad \epsilon = 0.$$

This can be written

$$I'(\epsilon) = \frac{d}{d\epsilon}\int_a^b F(x, f + \epsilon\eta, f' + \epsilon\eta')\,dx = \int_a^b \frac{\partial}{\partial\epsilon} F(x, f + \epsilon\eta, f' + \epsilon\eta')\,dx = 0.$$

We must now use the chain rule to differeniate F with respect to ϵ, Writing $y = f + \epsilon\eta$, $z = y' = f' + \epsilon\eta'$

$$\frac{\partial}{\partial\epsilon}F(x, y, z) = \frac{\partial F}{\partial y}\frac{\partial y}{\partial\epsilon} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\epsilon} = \frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial z}\eta'(x).$$

Evaluating this at $\epsilon = 0$, we have $y = f$ and $z = y' = f'$, so

$$\left[\frac{\partial}{\partial\epsilon}F(x, y, z)\right]_{\epsilon=0} = \frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x), \quad \text{with}\ \ y(x) = f(x),\ y'(x) = f'(x).$$

This means that the condition that $f(x)$ is an extremal is equivalent to

$$I'(0) = \int_a^b \frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\eta'(x)\,dx = 0. \quad \text{at}\ y = f,\ y' = f'. \qquad (*)$$

Integrating the second term by parts

$$\int_a^b \frac{\partial F}{\partial y'}\eta'(x)\,dx = \left[\frac{\partial F}{\partial y'}\eta(x)\right]_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\,dx.$$

The first term on the r.h.s. vanishes because $\eta(a) = \eta(b) = 0$ and on inserting the second term into (*) we get

$$\int_a^b \left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right\} \eta(x)\ dx = 0.$$

This equation is true *for any possible choice of* $\eta(x)$, so it must be case that the extremal $y(x) = f(x)$ satisfies

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0, \quad \text{with} \ \ y(a) = A, \ y(b) = B.$$

Example 1: Find a function to minimize the functional

$$I[y] = \int_0^1 (y' - y)^2\ dx, \qquad y(0) = 0,\ y(1) = 2, \qquad \left[\text{Answer:}\ \ y = f(x) = 2\frac{\sinh x}{\sinh 1}\right].$$

Example 2: Prove that the straight line $y = x$ is the curve giving the shortest distance between the points $(0, 0)$ and $(1, 1)$.

Example 3: Find an extremal function of

$$I[y] = \int_1^2 x^2 (y')^2 + y\ dx, \qquad y(1) = 1,\ y(2) = 1, \qquad \left[\text{Answer:}\ \ y = f(x) = \frac{1}{2}\ln x + \frac{\ln 2}{x} + 1 - \ln 2\right].$$

The Beltrami Identity: Special case $F = F(y, y')$.

A necessary condition for $y(x) = f(x)$ to be an extremal of a functional with an integrand *THAT DOES NOT DEPEND EXPLICITLY ON* x, i.e.

$$I = \int_a^b F(y, y')\, dx, \quad \text{subject to} \quad y(a) = A,\ y(b) = B,$$

is that $y(x) = f(x)$ satifies the first order ordinary differential equation defined by

$$F - y'\frac{\partial F}{\partial y'} = C \quad \text{(constant), with } \ y(a) = A,\ y(b) = B,$$

where the 'extra' boundary condition on $y(x)$ is used to determine the otherwise unknown constant C.

This is the *Beltrami Identity*, and yields an equation that is usually, but not always, easier to solve than that obtained directly from Euler's equation.

Proof:

Consider

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{dF}{dx} - y''\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right).$$

Differentiating $F(x, y(x), y'(x))$ **explicitly** with respect to x we get that

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} - y''\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

Noting that $\partial F/\partial x = 0$, as $F = F(y, y')$, we are left with

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) = 0 \ \text{ for } y(x) = f(x),$$

because $y(x) = f(x)$ satisfies Euler's equation. The left hand side can now be integrated with respect to x to reveal that $y(x) = f(x)$ must be a solution of

$$F - y'\frac{\partial F}{\partial y'} = C \quad \text{(constant), with } \ y(a) = A,\ y(b) = y(B),$$

for some unknown constant C.

Note: In the more straightforward special case $F = F(y')$ (covered in lectures), the extremal is always a straight line,

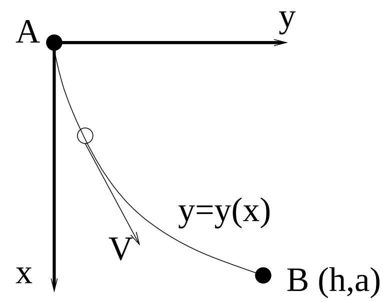


Figure: Illustrating the brachistochrone problem.

The Brachistochrone (Example 4)

A classic early application of the calculus of variations was to find *the brachistochrone*, defined as that smooth curve joining two points A and B (not underneath one another) along which a particle will slide from A to B under gravity in the fastest possible time.

Using the coordinate system illustrated above, we can use conservation of energy to obtain the velocity v of the particle as it makes its descent

$$\frac{1}{2}mv^2 = mgx, \qquad v = \sqrt{2gx}.$$

Noting also that distance along the curve s satisfies $ds^2 = dx^2 + dy^2$, we can express the time taken for the particle to descend as a functional of the shape of the curve $y(x)$,

$$T = \int_A^B dt = \int_A^B \frac{ds}{ds/dt} = \int_A^B \frac{ds}{v} = \int_0^h \frac{\sqrt{1+(y')^2}}{\sqrt{2gx}}\, dx, \quad \text{subject to} \quad y(0) = 0,\ y(h) = a.$$

The brachistochrone is an extremal of this functional, so we may use Euler's equation to get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{2gx(1+(y')^2)}}\right) = 0, \quad y(0) = 0,\ y(h) = a.$$

Integrating this, get

$$\frac{y'}{\sqrt{2gx(1+(y')^2)}} = c \ \ \text{(constant)},$$

and rearranging

$$y' = \frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{\alpha - x}}, \qquad \text{with} \ \ \alpha = \frac{1}{2gc^2}.$$

We can integrate this equation using the substitution $x = \alpha \sin^2\theta$ to obtain

$$y = \int \frac{\sqrt{x}}{\sqrt{\alpha - x}}\, dx = \int \frac{\sin\theta}{\cos\theta} 2\sqrt{\alpha}\sin\theta\cos\theta\, d\theta = \int \sqrt{\alpha}(1-\cos 2\theta)\, d\theta = \frac{\sqrt{\alpha}}{2}(2\theta - \sin 2\theta) + k.$$

Substituting back for x get

$$y(x) = \alpha \sin^{-1}\sqrt{\frac{x}{\alpha}} - \sqrt{x}\sqrt{\alpha - x}.$$

This curve is a *cycloid.*

It can be constructed by following the locus of the initial point of contact when a circle of radius $\alpha/2$ is rolled along a straight line (see lectures.)

Constraints in the Calculus of Variations - Isoperimetric Problems

Constraints may also enter problems in the calculus of variations, see for example the classic 'hanging chain' problem (sheet 4, Q1). Generally, constraints are dealt with using Lagrange multipliers, in a similar fashion to that used in §1.

A typical calculus of variations problem with constraint is to find the extremal of

$$I = \int_a^b F(x, y, y')\, dx, \quad \text{with } y(a) = A,\ y(b) = B, \quad \text{subject to} \quad J = \int_a^b G(x, y, y')\, dx = C.$$

The condition $J = C$ (constant) is the constraint.

Explicitly, a necessary condition for $y(x) = f(x)$ to be the extremal of I subject to $J = C$ is that $y(x) = f(x)$ is an extremal of

$$K = \int_a^b F(x, y, y') + \lambda G(x, y, y')\, dx, \quad \text{with } y(a) = A,\ y(b) = B.$$

where the constant Lagrange multiplier λ may be determined using the constraint $J = C$.

Proof:

Let y(x)=f(x) be the extremal of I subject to $J = C$, (with $f(a) = A$, $f(b) = B$), and consider the two-parameter family of admissible functions given by

$$y(x) = f(x) + \epsilon\eta(x) + \delta\zeta(x). \quad \text{(satisfying } \eta(a) = \zeta(a) = \eta(b) = \zeta(b) = 0.)$$

$\eta(x)$ and $\zeta(x)$ are arbitrary twice differentiable functions (but must be independent) except for satisfying the end conditions, and ϵ and δ are parameters.

Next, consider the functions of two variables

$$I(\epsilon, \delta) = \int_a^b F(x, f + \epsilon\eta + \delta\zeta, f' + \epsilon\eta' + \delta\zeta')\, dx, \quad J(\epsilon, \delta) = \int_a^b G(x, f + \epsilon\eta + \delta\zeta, f' + \epsilon\eta' + \delta\zeta')\, dx.$$

Because $y(x) = f(x)$ is an extremal of I subject to $J = C$, it follows that at the point (ϵ, δ)=$(0, 0)$, $I(\epsilon, \delta)$ must take an extreme value subject to $J(\epsilon, \delta) = C$.

From §1 of the course, a necessary condition for a function of two variables $I(\epsilon, \delta)$ subject to a constraint $J(\epsilon, \delta) - C = 0$ to take an extreme value at a particular point $(0, 0)$ is that $(0, 0)$ is a critical point of

$$K(\epsilon, \delta) = I(\epsilon, \delta) + \lambda J(\epsilon, \delta),$$

or that

$$\frac{\partial}{\partial \epsilon}\left[I(\epsilon, \delta) + \lambda J(\epsilon, \delta),\right]\Big|_{\epsilon=\delta=0} = \frac{\partial}{\partial \delta}\left[I(\epsilon, \delta) + \lambda J(\epsilon, \delta)\right]\Big|_{\epsilon=\delta=0} = 0.$$

where λ is an initially unknown multiplier to be determined.

Expanding the function $K(\epsilon, \delta)$

$$\begin{aligned}
\frac{\partial}{\partial \epsilon}\left[I(\epsilon, \delta) + \lambda J(\epsilon, \delta)\right] &= \int_a^b \frac{\partial}{\partial \epsilon}\left(F(x, f + \epsilon\eta + \delta\zeta, f' + \epsilon\eta' + \delta\zeta') + \lambda G(x, f + \epsilon\eta + \delta\zeta, f' + \epsilon\eta' + \delta\zeta')\right) dx \\
&= \int_a^b \eta\frac{\partial}{\partial y}(F + \lambda G) + \eta'\frac{\partial}{\partial y'}(F + \lambda G)\, dx \qquad \text{(chain rule)} \\
&= \int_a^b \eta\left\{\frac{\partial}{\partial y}(F + \lambda G) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(F + \lambda G)\right)\right\} dx \qquad \text{(integration by parts)} \\
&= 0, \qquad \text{at } (\epsilon, \delta) = (0, 0).
\end{aligned}$$

The above is true for an arbitrary function $\eta(x)$, so the terms inside the brackets must be identically zero, i.e.

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}(F+\lambda G)\right) - \frac{\partial}{\partial y}(F+\lambda G) = 0, \qquad \text{at } \epsilon = \delta = 0.$$

Evaluation at $(\epsilon, \delta) = (0,0)$ is equivalent to evaluation at $y = f(x)$, $y' = f'(x)$, so $y = f(x)$ satisfies Euler's equation for the functional K as postulated.

Note that the initially unknown multiplier λ must be determined using the constraint $J = C$.

Example 5: Minimize the functional

$$I[y] = \int_0^1 (y')^2 \, dx, \qquad y(0) = y(1) = 1,$$

subject to the constraint that

$$J[y] = \int_0^1 y \, dx = 2. \qquad \left[\text{Answer: } y = f(x) = -6\left(x - \frac{1}{2}\right)^2 + \frac{5}{2}.\right].$$

Example 6: Sheep pen design problem: A fence of length l must be attached to a straight wall at points A and B (a distance a apart, where $a < l$) to form an enclosure. Show that the shape of the fence that maximizes the area enclosed is the arc of a circle, and write down (but do not try to solve) the equations that determine the circle's radius and the location of its centre in terms of a and l.

MATH2401 §3 First-Order Partial Differential Equations

Partial Differential Equations: Some Definitions

A **partial differential equation** (p.d.e.) is a differential equation involving more than one independent variable.

$$\text{E.g.}\quad \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0,\quad \text{(Laplace's equation)}.$$

Here z is the dependent variable and x and y are the independent variables.

The **order** of a p.d.e. is the *highest* order partial derivative appearing.

The **degree** of a p.d.e. is the highest power or product of the dependent variable z or its partial derivatives.

First degree p.d.e.s are also called **linear**.

A p.d.e. is called **quasi-linear** if the highest derivative appears only in the first degree.

Examples:

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}-\frac{\partial^2 z}{\partial x^2} &= 0, &&\text{second order, first degree, linear (1d wave equation).}\\
z^4\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial y}\right)^4 &= 0, &&\text{first order, fifth degree}\\
z\frac{\partial^2 z}{\partial x\partial y}+\left(\frac{\partial z}{\partial y}\right)^3 &= 0, &&\text{second order, third degree, quasi-linear}
\end{aligned}$$

If there is only one independent variable, the differential equation is **ordinary** (o.d.e.).

The general solution of an o.d.e. involves arbitrary constants, whereas the general solution of a p.d.e. involves *arbitrary functions.*

Example 1: Consider the linear equation for $z=z(x,y)$,

$$\frac{\partial z}{\partial x}+z=x.\qquad (*)$$

Just as with o.d.e.s, linear equations in this form can be solved by finding the **complementary function** and a **particular integral**. The complementary function (C.F.) is the general solution of the homogeneous equation

$$\frac{\partial z}{\partial x}+z=0,$$

whereas the particular integral (P.I.) is **any** solution of the full equation $(*)$. The general solution of $(*)$ is (see lecture notes)

$$z(x,y)=\underbrace{f(y)e^{-x}}_{\text{C.F.}}+\underbrace{x-1}_{\text{P.I.}}.$$

Linear First-Order PDEs with Constant Coefficients

First we aim to find general solutions $z = z(x, y)$ of equations of the form

$$A\frac{\partial z}{\partial x} + B\frac{\partial z}{\partial y} + Cz = G(x, y), \qquad (*)$$

with $A \neq 0$, B, C constants and $G(x, y)$ an arbitrary function.

First consider the homogenous equation

$$A\frac{\partial z}{\partial x} + B\frac{\partial z}{\partial y} + Cz = 0.$$

Define

$$\zeta = ax + by, \quad \text{and} \quad \eta = cx + dy, \quad \text{with} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Using the chain rule we can write

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial x} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial x} &= az_\zeta + cz_\eta, \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial \zeta}\frac{\partial \zeta}{\partial y} + \frac{\partial z}{\partial \eta}\frac{\partial \eta}{\partial y} &= bz_\zeta + dz_\eta,\end{aligned}$$

so inserting these into $(*)$ we have

$$(Aa + Bb)z_\zeta + (Ac + Bd)z_\eta + Cz = 0.$$

This suggests a choice such as $a = 1$, $b = 0$, $c = B$, $d = -A$, giving

$$Az_\zeta + Cz = 0, \quad \text{with} \quad \zeta = x, \quad \eta = Bx - Ay, \qquad (**)$$

which can be integrated to give the complementary function

$$z = f(\eta)e^{-C\zeta/A} = f(Bx - Ay)e^{-Cx/A}.$$

Example 2: Find the general solution of

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = 0, \qquad [\text{Answer: } z = f(x - y)e^{-x}].$$

Particular Integrals: To find the particular integral, as with o.d.e.s it is possible to use a range of techniques, including simply looking for solutions using educated guesses. A fairly general method is to use an integrating factor, exactly as with o.d.e.s, by writing the full equation, based on $(**)$, in the form

$$\frac{\partial}{\partial \zeta}\left(Aze^{C\zeta/A}\right) = e^{C\zeta/A}G(x, y) = e^{C\zeta/A}G(\zeta, \frac{1}{A}(B\zeta - \eta)).$$

The right-hand side may be integrated with respect to ζ, treating η as a constant.

Example 3: Find the general solution of

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = x + y, \qquad [\text{Answer: } z = f(x - y)e^{-x} + x + y - 2].$$

Example 4: Find the general solution of

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + z = \sinh\{3x + y\}, \qquad [\text{Answer: } z = f(x - y)e^{-x} + \frac{4}{15}\cosh\{3x + y\} - \frac{1}{15}\sinh\{3x + y\}.].$$

Linear First-Order PDEs with Non-constant Coefficients

Next, we consider more general linear equations of the form

$$A(x,y)\frac{\partial z}{\partial x} + B(x,y)\frac{\partial z}{\partial y} + C(x,y)z = G(x,y), \qquad (*)$$

(note that the constant coefficient equations above are a special case of $(*)$.) If we are to find the general solution in a similar fashion to that used above we will have to generalise our co-ordinate transform by allowing the transformed variables to depend on x and y in a general way,

$$\zeta = \zeta(x,y), \quad \text{and} \quad \eta = \eta(x,y).$$

Using the chain rule

$$z_x = z_\zeta\,\zeta_x + z_\eta\,\eta_x, \qquad z_y = z_\zeta\,\zeta_y + z_\eta\,\eta_y.$$

Substituting into $(**)$ get

$$(A\zeta_x + B\zeta_y)z_\zeta + (A\eta_x + B\eta_y)z_\eta + Cz = G. \quad (*)$$

Following the constant coefficients equation above, we'd like to choose $\eta(x,y)$ so that

$$A(x,y)\eta_x + B(x,y)\eta_y = 0.$$

Consider the O.D.E. given by

$$\frac{dy}{dx} = \frac{B(x,y)}{A(x,y)},$$

the solutions of this equation are curves that may be written $f(x,y) = c$, where c is a constant. Taking the full derivative with respect to x along these curves

$$\begin{aligned} \frac{d}{dx}f(x,y(x)) = f_x + \frac{dy}{dx}f_y &= 0, \qquad \text{(chain rule)} \\ = f_x + \frac{B}{A}f_y &= 0, \qquad \text{(from the O.D.E).} \end{aligned}$$

Therefore $A(x,y)f_x + B(x,y)f_y = 0$, and $\eta = f(x,y)$ is the desired choice of variable.

Definition: The variable $\eta = f(x,y)$ is known as the *charactersitic variable.*

Definition: The curves defined by $\eta = f(x,y)$ =cons. are known as *charactersitic projections.*

Inserting $\zeta = x$ and $\eta = f(x,y)$ into $(*)$,

$$\tilde{A}(\zeta,\eta)\frac{\partial z}{\partial \zeta} + \tilde{C}(\zeta,\eta)z = \tilde{G}(\zeta,\eta),$$

where $\tilde{A}(\zeta,\eta) = A(x(\zeta,\eta), y(\zeta,\eta))$ etc. This equation can be integrated like an O.D.E., treating η as a constant, provided an *arbitrary function of integration* is used in place of a constant. For the homogeneous equation ($G = \tilde{G} = 0$) the general solution is

$$z(\zeta,\eta) = f(\eta)\exp\left\{-\int \frac{\tilde{C}(\zeta,\eta)}{\tilde{A}(\zeta,\eta)}d\zeta\right\}.$$

Finding Particular Integrals: Note that non-zero $G(x,y)$ can be dealt with exactly as described for the constant coefficient equations.

Example 5: Find the general solution of

$$x\frac{\partial z}{\partial x} - 7y\frac{\partial z}{\partial y} = x^2 y, \qquad \text{[Answer: } f(x^7 y) - \frac{1}{5}x^2 y.\text{].}$$

Example 6: Find the general solution of

$$x^2\frac{\partial z}{\partial x} + -xy\frac{\partial z}{\partial y} + yz = 0, \qquad \text{[Answer: } z = f(xy)\exp\left\{\frac{y}{2x}\right\}\text{].}$$

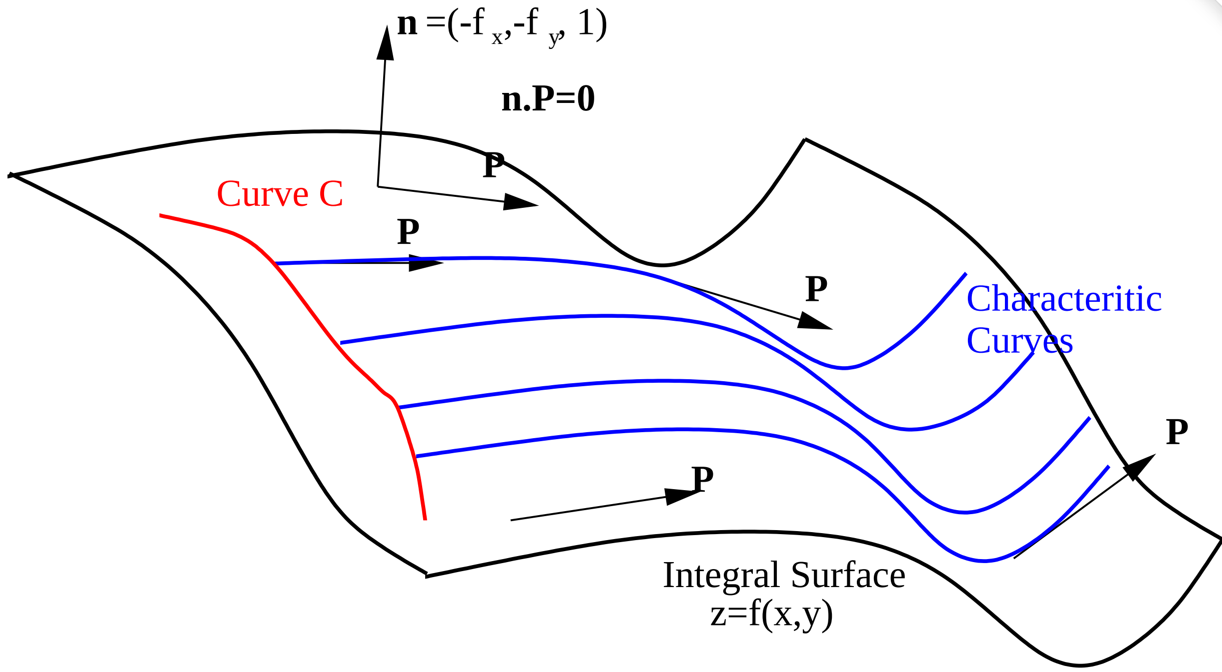


Figure: A geometrical intepretation of the method of Lagrange.

Method of Lagrange: First-Order Quasi-Linear Equations

The method of Lagrange is used to solve *quasi-linear equations* of the form

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z), \qquad (\dagger).$$

Note that the linear equations above are special cases of (†) with $P = A$, $Q = B$, $R = G - Cz$, hence the method of Lagrange can also be used with them.

The method of Lagrange is based on the observation that solutions of (†) can be generated from solutions of the *associated ODEs*

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad (\ddagger)$$

Definition: The solutions of (†) are surfaces, and are known as *integral surfaces.*
Definition: The solutions of (‡) are curves known as *characteristic curves.*

If the characteristic curves have equation of the form $u(x,y,z) = c$, $v(x,y,z) = d$ (a curve is the intersection of two surfaces), then the general solution of the original equation (†) is

$$F(u,v) = 0, \qquad \text{(or equivalently } u = f_1(v), \text{ or } v = f_2(u) \text{)},$$

for an arbitrary function F (or f_1, or f_2).

Proof of the Method of Lagrange

The method of Lagrange is perhaps best understood geometrically (see diagram).

Consider the three-dimensional vector field $\mathbf{P} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. The normal to a given integral surface $z = f(x, y)$ is perpendicular to $\mathbf{P}$ at every point on the surface, since the (upward pointing) normal $\mathbf{n}$ is given by

$$\mathbf{n} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) \quad \text{so that} \qquad \mathbf{n}.\mathbf{P} = -P\frac{\partial f}{\partial x} - Q\frac{\partial f}{\partial y} + R = 0,$$

and $z = f(x, y)$ is a solution of the equation (†).

The solutions of the associated ODEs (‡) (discussed below) are curves with equation $u(x, y, z) = c$, $v(x, y, z) = d$ that satisfy

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = d\lambda,$$

or

$$\frac{\mathbf{dx}}{d\lambda} = \mathbf{P}.$$

In other words the characteristics are curves $\mathbf{x} = \mathbf{x}(\lambda)$ that are everywhere *tangent* to the vector field $\mathbf{P}$.

Hence any surface generated from characteristic curves will have normal perpendicular to $\mathbf{P}$, and will therefore be an integral surface.

This also means that if a characteristic curve meets an integral surface $z = f(x, y)$ at any point, it must necessarily lie entirely within that surface (see diagram).

If a curve C (not a characteristic) lying in an integral surface has equation $\psi(x, y, z) = \phi(x, y, z) = 0$, (intersection of two surfaces) the equation of the entire surface can be obtained by finding the equation of all those characteristic curves that intersect C.

i.e. Eliminate x,y,z between $\psi(x, y, z) = 0$, $\phi(x, y, z) = 0$, $u(x, y, z) = c$ and $v(x, y, z) = d$.

Elimating 3 variables between 4 equations results in a relation between the two parameters c and d, of the form $F(c, d) = 0$. The equation of all those characteristics intersecting the curve C is therefore

$$F(u(x, y, z), v(x, y, z)) = 0. \quad \text{where } F \text{ is in general an arbitrary function.}$$

Example 7: Find the general solution of

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = z^2, \qquad [\text{Answer: } F\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{x}\right) = 0].$$

Solving the Associated ODEs

In practise the associated ODEs are usually formed from (‡) and solved in pairs, e.g. any two from

$$\frac{dy}{dx} = \frac{Q}{P}, \quad \frac{dz}{dx} = \frac{R}{P}, \quad \text{and} \quad \frac{dz}{dy} = \frac{R}{Q}.$$

thereby obtaining functions $u(x, y, z) = c$ and $v(x, y, z) = d$, with c and d being constants of integration.

In general, however, it is not usually possible to separate variables to solve the equations.

It is helpful sometimes to use the *componendo et dividendo* rule for fractions to write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{\lambda dx + \mu dy + \nu dz}{\lambda P + \mu Q + \nu R}, \qquad \text{for any } \lambda,\ \mu,\ \nu.$$

This can help in one (or both) of two ways

1. If we can find λ, μ, ν that allows us to form a separable equation.

2. If we can find λ, μ, ν so that $\lambda P + \mu Q + \nu R = 0$ together with some $u(x, y, z)$ that satisfies $\nabla u = (\lambda, \mu, \nu)$. Then, to avoid a contradiction we must have

$$\lambda dx + \mu dy + \nu dz = 0, \quad \text{or} \quad \nabla u.\mathbf{dx} = \mathbf{0},$$

   which may be integrated to give $u(x, y, z) = c$ (constant) along characteristics, giving one of the constants of integration we are seeking.

Example 8: Find the general solution of

$$(y - x)\frac{\partial z}{\partial x} + (y + x)\frac{\partial z}{\partial y} = \frac{x^2 + y^2}{z}, \qquad [\text{Answer: } F(\,(x + y)^2 - 2y^2,\ x^2 - y^2 + z^2\,) = 0.]$$

Example 9: Find the general solution of

$$x(y^2 - z^2)\frac{\partial z}{\partial x} + y(z^2 - x^2)\frac{\partial z}{\partial y} = z(x^2 - y^2), \qquad [\text{Answer: } F(\,xyz, x^2 + y^2 + z^2\,) = 0\ .]$$

Cauchy's Problem: Using Boundary Conditions to Find a Specific Soln.

As with ordinary differential equations, boundary data can be used to determine *specific solutions* from the *general solutions* we have derived so far. In other words, boundary conditions may be used to determine the unknown functions f in the general solutions.

For a first order ODE, the boundary condition is supplied at a single point, whereas for a PDE, the boundary condition is supplied on a curve (i.e. the curve C is specified in the 'method of Lagrange' diagram).

Example 10: Find the specific solution consistent with the initial data

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z, \quad z(1, y) = y. \quad \text{[Answer: } z = \tfrac{1}{2}(2y - 3x + 3)\exp\{(x-1)/2\}.\text{]}$$

Example 11: Find the specific solution passing through the given boundary curve C,

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} + z^2 = 0, \quad \text{with C defined by } xy = x + y, \ z = 1. \quad \text{[Answer: } z = 2xy/(3xy - x - y).\text{]}$$

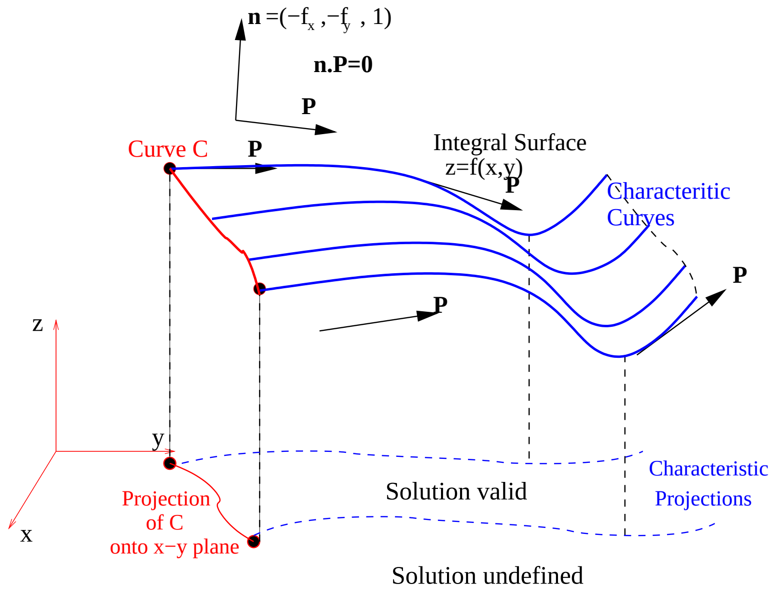


Figure: Geometrical view of the situation when initial data is given only a finite boundary curve C.

Cauchy's Problem with Finite Boundary Curves

Sometimes the boundary data for a PDE is given only on a finite curve. In this case, it is necessary to determine both the solution and its region of validity in the (x, y)-plane.

As an integral surface is constructed by extending characteristics outwards from the boundary curve (C in diagram). If C is finite the integral surface that is generated will also be finite in extent. To find the region of validity of the solution of the equation in the (x, y) plane we can consider the charactersitic projections of those characteristics that pass through the ends of C as shown.

Example 12 (10 revisited): Find the specific solution consistent with the initial data given on a finite curve. Indicate the region of validity of the solution in the (x, y) plane

$$2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = z, \quad z(1, y) = y. -1 \leq y \leq 1.$$

$$\left[\text{Answer: } z = \tfrac{1}{2}(2y - 3x + 3)\exp\{(x-1)/2\}, \text{ valid in } \quad 3x - 5 < 2y < 3x - 1.\right]$$

Example 13: Find the specific solution consistent with the initial data given on a finite curve. Indicate the region of validity of the solution in the (x, y) plane

$$e^x\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0, \quad z(x, 0) = \tanh x, 1 \leq x \leq 2.$$

$$[\text{Answer: } z = \tfrac{1-(y+e^{-x})^2}{1+(y+e^{-x})^2} \text{ valid in } e^{-2} - e^{-x} < y < e^{-1} - e^{-x}.]$$

MATH2401 §4 Second-Order PDEs: The Wave Equation

Second-Order PDEs: General Forms and Definitions

The general form for a quasi-linear second-order PDE with two independent variables x, y is

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = r, \qquad (*),$$

where a, b, c, and r are functions of x, y, z, $\partial z/\partial x$ and $\partial z/\partial y$. The equation $(*)$ is said to be *linear* if a, b, c are functions of x and y only, and

$$r = d(x,y)\frac{\partial z}{\partial x} + e(x,y)\frac{\partial z}{\partial y} + f(x,y)z + g(x,y).$$

Definition: The quantity $\Delta = b^2 - 4ac$ is the *discriminant* of $(*)$.
Definition: If $\Delta > 0$ then $(*)$ is said to be *hyperbolic*.
Definition: If $\Delta = 0$ then $(*)$ is said to be *parabolic*.
Definition: If $\Delta < 0$ then $(*)$ is said to be *elliptic*.

Second-Order PDEs with Constant Coefficients

Consider

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = 0, \qquad (a,\, b,\, c \text{ constants}),\ (**).$$

Seeking a solution of the form $z(x,y) = f(y + mx)$, the auxillary equation

$$am^2 + bm + c = 0,$$

is obtained. The form of the *general solution* of $(**)$ depends on the nature of the roots m_1 and m_2 of the auxillary equation.

Hyperbolic case: $\Delta = b^2 - 4ac > 0$, $m_1 \neq m_2$, **both real.**

The general solution is given by

$$z(x,y) = f(y + m_1 x) + g(y + m_2 x), \qquad \text{with } f,\ g \text{ arbitrary functions.}$$

Proof: Use the transformation to the *canonical variables* $s = y + m_1 x$, $t = y + m_2 x$ to show this explicitly (see notes).

Example 1: Find the solution of

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$$

[Answer: $z(x, y) = f(y + x) + g(y + 2x)$].

Parabolic case: **$\Delta = b^2 - 4ac = 0$, $m_1 = m_2$, real.**

The general solution is given by

$$z(x, y) = f(y + m_1 x) + xg(y + m_1 x), \qquad \text{with } f,\ g \text{ arbitrary functions.}$$

Proof: Use the transformation to the *canonical variables* $s = y + m_1 x$, $t = x$ to show this explicitly (see notes).

Example 2: Find the solution of

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

[Answer: $z(x, y) = f(y + x) + xg(y + x)$].

Elliptic case: **$\Delta = b^2 - 4ac < 0$, $m_1 = m_2^*$, complex conjugate roots.**

The general solution is again

$$z(x, y) = f(y + m_1 x) + g(y + m_2 x), \qquad \text{with } f,\ g \text{ arbitrary functions.}$$

Note that for this case, although m_1 and m_2 are complex, it is possible to construct real solutions by suitable (complex) choices of f and g.

Non-homogeneous equations: Particular Integrals

As with first order PDEs, two simple techniques for finding with particular integrals for non-homogeneneous equations (non-zero right hand side) are

- Try to 'spot' an appropriate solution. For example

  $$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{x-y}$$

  Try $z(x, y) = Ae^{x-y}$. (Insert in equation, get $A = 1/6$).

- Convert the equation to canonical variables (e.g. $s = y + m_1 x$, $t = y + m_2 x$ in the parabolic case) and then integrate as normal. In canonical form the example above becomes

  $$-\frac{\partial^2 z}{\partial s \partial t} = e^{2t-3s}$$

  which integrates to give $z = e^{2t-3s}/6 = e^{x-y}/6$ as above.

The first method is often faster, but the second is more systematic and can be used more generally.

The Wave Equation: D'Alembert's Solution

One of the most important **linear**, **hyperbolic** second-order PDEs, due to its appearance in many branches of physics, is the wave equation

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \qquad c \text{ constant} \qquad (\dagger).$$

For example, it will be shown that the wave equation describes the evolution of small amplitude transverse waves on a taut string, with $c = \sqrt{T/\rho}$, and T tension, ρ density. It also describes the evolution of small amplitude long waves in shallow water (tsunamis in the deep ocean!). In this case $c = \sqrt{gH}$ for g gravity, H water depth.

First, we will consider solutions of ($\dagger$) on the domain $-\infty < x < \infty$, valid for $t \geq 0$.

The general solution of ($\dagger$) is easily shown to be (try $z = f(x + mt)$ to yield the auxillary equation $m^2 = c^2$)

$$z(x,t) = f(x+ct) + g(x-ct), \qquad \text{with } f,\ g \text{ arbitrary functions.}$$

All solutions of ($\dagger$) can be written in the above form. However, for physical problems defined on the domain $-\infty < x < \infty$, it is often the *specific solution*, in terms of some *initial data* given at $t = 0$, that is required. Initial data is of the form

$$z(x,0) = F(x), \qquad \frac{\partial z}{\partial t}(x,0) = G(x).$$

Note that specifying initial data in the above form is analogous, for example, to specifying $y(0) = A$ and $y'(0) = B$ to obtain a specific solution of the ordinary differential equation $ay'' + by' + cy = 0$. Note also that for the problem of waves on a string (see later), $F(x)$ is physically the initial 'displacement' of the string, and $G(x)$ is its initial 'velocity' in the direction of increasing z.

To proceed, it is necessary to find f and g in terms of F and G. At $t = 0$, equating the general solution and initial conditions,

$$\begin{aligned} F(x) &= f(x) + g(x) \qquad (*) \\ G(x) &= c(f'(x) - g'(x)). \end{aligned}$$

Integrating the second of these with respect to x gives

$$\int_\alpha^x G(\zeta)\,d\zeta = c(f(x) - g(x)), \qquad \alpha \text{ arbitrary constant.}$$

which can be combined with $(*)$ to give

$$\begin{aligned} f(x) &= \frac{1}{2}F(x) + \frac{1}{2c}\int_\alpha^x G(\zeta)\,d\zeta, \\ g(x) &= \frac{1}{2}F(x) - \frac{1}{2c}\int_\alpha^x G(\zeta)\,d\zeta. \end{aligned}$$

The above expressions give f and g as functions of the (dummy) variable x. In the general solution of the wave equation above, however, f and g appear as functions of $x + ct$ and $x - ct$ respectively, giving,

$$z(x,t) = \frac{1}{2}\left(F(x+ct) + F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\zeta)\,d\zeta.$$

This is **D'Alembert's solution** of the wave equation with specified initial data.

Example 3: Find the solution of

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \quad \text{with} \quad z(x,0) = e^{-x^2}, \quad z_t(x,0) = 0.$$

$$\text{[Answer:} \quad z(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2} + e^{-(x-ct)^2}\right) \text{].}$$

Example 4: Find the solution of

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \quad \text{with} \quad z(x,0) = 0, \quad z_t(x,0) = \frac{1}{1+x^2}.$$

$$\text{[Answer:} \quad z(x,t) = \frac{1}{2c}\left(\tan^{-1}(x+ct) - \tan^{-1}(x+ct)\right) \text{].}$$

Sketch $z(x,t)$ as a function of x at $t = 1/c$ and for large t in each case.

The following examples are for restricted initial data (see over).

Example 5: Find the solution of

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} \quad \text{with} \quad z(x,0) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}, \quad z_t(x,0) = 0.$$

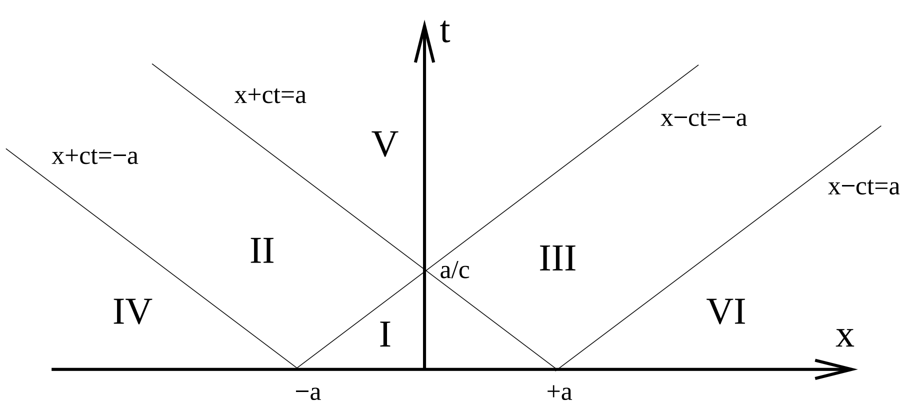
$$\text{[Answer:} \quad z(x,t) = \begin{cases} 1 & \text{in region I} \\ \frac{1}{2} & \text{in region II} \\ \frac{1}{2} & \text{in region III} \\ 0 & \text{in region IV, V, VI} \end{cases} \quad \text{].}$$

Example 6: Find the solution of

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \quad \text{with} \quad z(x,0) = 0, \quad z_t(x,0) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$$\text{[Answer:} \quad z(x,t) = \begin{cases} t & \text{in region I} \\ \frac{1}{2c}(x+ct+a) & \text{in region II} \\ \frac{1}{2c}(a-x+ct) & \text{in region III} \\ \frac{a}{c} & \text{in region V} \\ 0 & \text{in region IV, VI} \end{cases} \quad \text{].}$$

Sketch $z(x,t)$ as a function of x at $t = 1/2c$ and $t = 2/c$ in each case.



Illustrating the different solution regimes for D'Alembert's solution in the (x,t)-plane when the initial data is as described below.

D'Alembert's Solution for Localised Initial Data

To develop an understanding of how solutions of the wave equation evolve as time t increases, according to D'Alembert's solution, it is helpful to consider initial data that is non-zero on just a finite part of the spatial domain. For example,

$$z(x,0) = \begin{cases} F(x) & |x| \le a \\ 0 & |x| > a \end{cases} \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = \begin{cases} G(x) & |x| \le a \\ 0 & |x| > a. \end{cases}$$

For the above initial data, the solution takes a different form in different regions of the (x,t)-plane, as illustrated in the diagram. The reason for this is clear if the above initial conditions are inserted *carefully* into D'Alembert's solution, noting that $F(x+ct)$ is non-zero only in $|x+ct| \le a$, $F(x-ct)$ is non-zero only in $|x-ct| \le a$, and that only values of $\zeta \in [-a,a]$ can contribute to the integral of G. Hence if, for example, $x+ct > a$, the upper limit of integration can be replaced by a.

Putting all of this together, the following is obtained,

$$z(x,t) = \begin{cases} \frac{1}{2}\left(F(x+ct)+F(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\zeta)\,d\zeta & \text{in region I, } |x+ct| \le a, |x-ct| \le a, \\ \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{-a}^{x+ct} G(\zeta)\,d\zeta & \text{in region II, } |x+ct| \le a, x-ct < -a, \\ \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{a} G(\zeta)\,d\zeta & \text{in region III, } x+ct > a, |x-ct| \le -a. \\ 0 & \text{in region IV, } x+ct < -a, x-ct < -a, \\ \frac{1}{2c}\int_{-a}^{a} G(\zeta)\,d\zeta. & \text{in region V, } x+ct > a, x-ct < -a. \\ 0 & \text{in regions IV, } x+ct > a, x-ct > a. \end{cases}$$

The above solution illustrates some important properties of the solutions of the wave equation:

- Information propagates both leftwards along characteristics $x+ct =$ cons., and rightwards along characteristics $x-ct =$ cons. as time increases.
- Hence the solution only evolves in time within the 'domain of influence' of the initial conditions: illustrated by regions I, II and III in the diagram, which is bounded by the characteristics emanating from $x = \pm a$.
- Note that an initial disturbance $z(x,0) = F(x)$ divides into two equal 'waves' one travelling leftwards with speed c, one travelling rightwards with speed c.

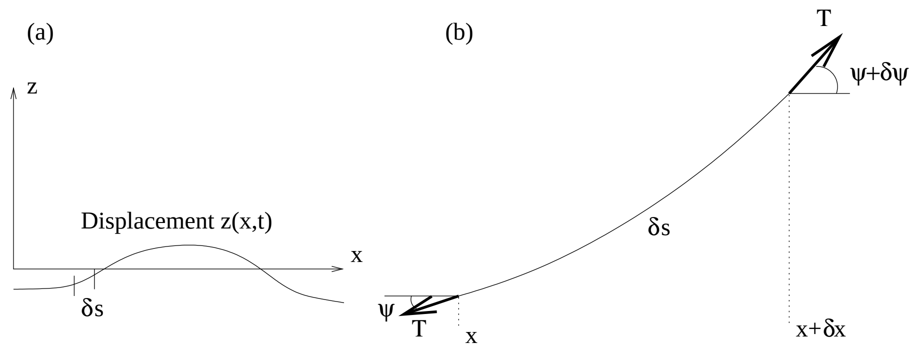


Figure: Showing (a) small amplitude waves on a string, and (b) the forces acting on a small section of the string.

Some Physics: Small Amplitude Waves on a String

Consider a small section of string with constant density ρ and under constant tension T as illustrated. The displacement of the string in the vertical is $z(x,t)$. Resolving forces in the vertical direction, Newton's second law (Force = mass × acceleration) gives

$$\underbrace{\rho\,\delta s}_{\text{Mass}} \times \underbrace{\frac{\partial^2 z}{\partial t^2}}_{\text{Acceleration}} = \underbrace{T\sin(\psi+\delta\psi) - T\sin\psi}_{\text{Forces}} \qquad = 2T\cos\left(\psi + \frac{\delta\psi}{2}\right)\sin\frac{\delta\psi}{2}$$

Taking $\delta\psi$ to be small,

$$\sin\frac{\delta\psi}{2} = \frac{\delta\psi}{2} - \frac{1}{6}\left(\frac{\delta\psi}{2}\right)^3 + \ldots.$$

$$\cos\left(\psi + \frac{\delta\psi}{2}\right) = \cos\psi\cos\frac{\delta\psi}{2} - \sin\psi\sin\frac{\delta\psi}{2} = \cos\psi - \frac{\delta\psi}{2}\sin\psi + \ldots.$$

Ignoring terms in $\delta\psi^2$ and higher gives

$$\rho\frac{\partial^2 z}{\partial t^2} = T\cos\psi\frac{\delta\psi}{\delta s},$$

We know that $\partial z/\partial x = \tan\psi$, so using the chain rule

$$\frac{\partial}{\partial s}\tan\psi = \sec^2\psi\frac{\partial\psi}{\partial s}, \quad \text{and} \quad \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} = \frac{\partial^2 z}{\partial x^2}\cos\psi, \quad \text{hence} \quad \frac{\partial\psi}{\partial s} = \cos^3\psi\frac{\partial^2 z}{\partial x^2}.$$

Since $\delta\psi/\delta s \to \partial\psi/\partial s$ for small δs, we have

$$\rho\frac{\partial^2 z}{\partial t^2} = T\cos^4\psi\frac{\partial^2 z}{\partial x^2}.$$

Finally, the fact that we are interested in **small amplitude** waves can be used. For small amplitude waves $\partial z/\partial x = \tan\psi \ll 1$, so using $\sec^2\psi = 1 + \tan^2\psi$ gives

$$\rho\frac{\partial^2 z}{\partial t^2} = T(1+\tan^2\psi)^{-2}\frac{\partial^2 z}{\partial x^2} \approx T\frac{\partial^2 z}{\partial x^2},$$

or to a good approximation

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \quad \text{with} \quad c = \sqrt{\frac{T}{\rho}}.$$

Hence small amplitude waves on a string satisfy the wave equation (to a good approximation), with the wave speed c depending on the tension T and the density ρ as above.

Simple Harmonic Waves and Complex Form

A rightwards-travelling simple harmonic wave is a solution to the wave equation of the form

$$z(x,t) = a\cos\left(\, k(x-ct) + \epsilon\,\right).$$

There are numerous associated definitions: a is the **amplitude**, k is the **angular wavenumber**, $\lambda = 2\pi/k$ is the **wavelength**, $\omega = kc$ is the **angular frequency** and $T = 2\pi/\omega$ is the **period**. If we have two, otherwise identical, waves with $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$ respectively, then $\epsilon_2 - \epsilon_1$ is known as the **phase shift** between the two waves.

Another special solution is obtained by adding a leftwards and a rightwards-travelling simple harmonic wave with equal amplitudes and angular wavenumbers,

$$\begin{aligned} z(x,t) &= a\cos\left(\, k(x+ct)\,\right) + a\cos\left(\, k(x-ct)\,\right) \\ &= 2a\cos\left(\, kx\,\right)\,\cos\left(\, kct\,\right) \end{aligned}$$

This solution is known as a **standing wave**, as it is not travelling to the left or right. Note that it is in *variables separable form* (see below).

It is often useful to write wave-like solutions of the wave equation in *complex form*. For example, the rightwards travelling simple harmonic wave above may be written

$$z(x,t) = Ae^{ik(x-ct)},$$

where it is understood that the real part is taken. The complex constant $A = ae^{i\epsilon}$ is known as a **complex amplitude** and contains information about the phase of the wave ϵ as well as its (physical) amplitude a. Note that $a = |A|$ and $\epsilon = \mathrm{Arg}\{A\}$.

Complex form is particularly convenient for problems where a travelling wave (on a string, say) is *incident* on a point (usually taken to be $x = 0$) where there is a physical discontinuity of some kind. For example, a mass could be attached to the string at $x = 0$, or two strings of different densities, and hence wave speeds, could be attached at $x = 0$ (see examples). For the case of a rightwards travelling wave incident from $x = -\infty$, we look for simple harmonic travelling wave solutions of the form

$$\begin{aligned} z_1(x,t) &= z_I + z_R &&= Ae^{i(\omega t - kx)} + Re^{i(\omega t + kx)} \quad &&\text{in} \quad x < 0, \\ z_2(x,t) &= z_S &&= Se^{i(\omega_2 t - k_2 x)} &&\text{in} \quad x > 0. \end{aligned}$$

Here z_I is the original *incident* wave, z_R is the *reflected* wave (due to reflection by the discontinuity at $x = 0$), and z_S is the *transmitted* wave that propagates into $x > 0$. The boundary conditions at the discontinuity $x = 0$ can be used to to obtain the complex amplitudes S and R in terms of A. The details, of course, depend on exactly what the boundary conditions are at $x = 0$. However continuity of displacement and velocity at $x = 0$ applies for most problems, giving

$$z_1(0,t) = z_2(0,t), \quad \frac{\partial z_1}{\partial t}(0,t) = \frac{\partial z_2}{\partial t}(0,t), \qquad \text{yielding} \qquad A + R = S, \quad \text{and} \quad \omega_2 = \omega.$$

The remaining boundary condition at $x = 0$ can then be used to solve the problem completely.

Example 7: Two semi-infinite strings with densities ρ_1, ρ_2 are joined at $x = 0$ and held under tension T. Small amplitude waves, with amplitude a propagate rightwards from $x = -\infty$ and are incident on $x = 0$. Find the amplitude of the reflected and transmitted wavetrains in terms of a.

$$\text{[Answer: } R = \left(\frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right) a, \quad S = \left(\frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}\right) a, \text{].}$$

Example 8: A mass M is attached at a point ($x = 0$) to an infinite string with density ρ and under tension T. Small amplitude waves, with amplitude a and angular wavenumber k propagate rightwards from $x = -\infty$ and are incident on $x = 0$. Find the complex amplitude of the reflected and transmitted wavetrains in terms of a.

$$\text{[Answer: } R = \left(\frac{1 - ip}{1 + p^2}\right) a, \quad S = \left(\frac{p^2 - ip}{1 + p^2}\right) a, \quad \text{where } p = \frac{Mk}{2\rho}\text{].}$$

The following examples require the separation of variables method (see over):

Example 9a (Plucked string): A string of density ρ, under tension T and of length L has fixed endpoints as in Example 9 (see over). The string is released from rest with an initial displacement given by

$$z(x, 0) = \begin{cases} \frac{2hx}{L} & 0 < x < L/2 \\ \frac{2h(L-x)}{L} & L/2 < x < L \end{cases}$$

Find an expression $z(x, t)$ for the subsequent evolution of the displacement.

$$\text{[Answer: } z(x,t) = \frac{8h}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \sin \frac{(2m+1)\pi x}{L} \cos \frac{(2m+1)\pi ct}{L}\text{].}$$

Example 10: A string of density ρ, under tension T and of length L is attached to light rings that are free to move along a smooth wire. At $t = 0$, $z(x, 0) = F(x)$, $z_t(x, 0) = G(x)$, $(0 \leq x \leq L)$. Find $z(x, t)$.

$$\text{[Answer: } z(x,t) = \sum_{m=0}^{\infty} \cos \frac{(2m+1)\pi x}{2L} \left[C_m \cos \frac{(2m+1)\pi ct}{2L} + D_m \sin \frac{(2m+1)\pi ct}{2L}\right],$$

$$\text{where} \quad C_m = \frac{2}{L} \int_0^L F(x) \cos \frac{(2m+1)\pi x}{2L}\, dx, \quad D_m = \frac{4}{\pi c(2m+1)} \int_0^L G(x) \cos \frac{(2m+1)\pi x}{2L}\, dx.\text{].}$$

Separation of Variables Method and Normal Modes

Up to this point, we have examined various solutions of the wave equation on the infinite domain $-\infty < x < \infty$. What about the situation when the domain is finite (e.g. $0 < x < L$), with boundary conditions given at $x = 0, L$?

To deal with this type of problem a new technique must be introduced; the method of separation of variables. A solution is sought in *variables separable form*

$$z(x,t) = X(x)T(t),$$

i.e. a solution which is the product of a function of x only and a function of t only. Inserting this in the wave equation

$$\frac{X}{c^2}\frac{d^2T}{dt^2} = T\frac{d^2X}{dx^2}, \qquad \text{or, dividing by XT,} \qquad \frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda, \quad \text{(constant).}$$

Note that λ must be a constant as we have equality between a function of t only and a function of x only (think about what happens when x and t are varied independently). There are three possibilities for λ:

Case I. $\lambda = p^2 > 0$:

$$X(x) = A\cosh px + B\sinh px, \qquad T(t) = C\cosh pct + D\sinh pct,$$

Case II. $\lambda = 0$:

$$X(x) = Ax + B, \qquad T(t) = Ct + D,$$

Case III. $\lambda = -p^2 < 0$:

$$X(x) = A\cos px + B\sin px, \qquad T(t) = C\cos pct + D\sin pct.$$

The solutions for Cases I and II are clearly unphysical for most problems, and nearly all sensible boundary conditions will lead to $A = B = 0$. Case III provides the wave-like solutions of interest.

Example 9: Consider waves on a string fixed at its endpoints $x = 0, L$, with initial displacement and velocity given by

$$z(x,0) = F(x), \qquad \frac{\partial z}{\partial t}(x,0) = G(x), \qquad 0 \le x \le L.$$

The endpoint conditions $z(0,t) = z(L,t) = 0$ provide boundary conditions for $X(x)$, namely $X(0) = X(L) = 0$. Applying these conditions, case III provides non-trivial solution with $A = 0$ and $p = n\pi/L$, where $n = 0, 1, 2, 3, \ldots$.. Without loss of generality we can set $B = 1$ (as we still have unknown constants C, D.) The solution must therefore consist of a linear combination of all possible case III solutions

$$z(x,t) = \sum_{n=1}^{\infty} z_n(x,t) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi ct}{L} + D_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}.$$

Definition: Each term in the summation $z_n(x,t)$ is known as a **normal mode** and has a corresponding angular frequency ω_n $(= n\pi c/L$ in this case).

Definition: The $n = 1$ mode is termed the **fundamental mode**.

Definition: The modes with $n = 2$, $n = 3$,.. are known as the **second harmonic**, **third harmonic**....etc.

The constants C_n, D_n can now be obtained from the initial conditions at $t = 0$. These give

$$F(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L}, \qquad G(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin \frac{n\pi x}{L}.$$

The standard Fourier series formulae result in

$$C_n = \frac{2}{L}\int_0^L F(x) \sin \frac{n\pi x}{L}\, dx, \qquad D_n = \frac{2}{n\pi c}\int_0^L G(x) \sin \frac{n\pi x}{L}\, dx.$$

In more complicated situations normal modes frequencies may be more difficult to determine:

Example 11: **A normal modes example**: Two identical strings of length L and density ρ, are knotted together at $x = 0$, with the knot having mass M. The other ends are attached to points $2L$ apart so that each string is under tension T. Small transverse disturbances in each string have amplitude $z_1(x,t)$ (in $-L \le x \le 0$) and z_2 (in $0 \le x \le L$) respectively, and one of the boundary conditions at $x = 0$ can be shown to be

$$M\frac{\partial^2 z_1}{\partial t^2} = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right).$$

Show, by writing the *time dependent* part of the disturbance in complex form, or otherwise, that the symmetric normal modes of vibration (about $x = 0$) of the string have period $2\pi/\omega$, where ω satisfies

$$\frac{\omega L}{c} \tan\left(\frac{\omega L}{c}\right) = \frac{2\rho L}{M},$$

and c is the wave speed in the string. What values does ω take when $M \to 0$? When $M \to \infty$? What are the frequencies associated with the antisymmetric modes of vibration? [**Answer**: $\omega = n\pi c/L$ $n = 1, 2, 3, ...$]

MATH2401 §5: The Heat Equation and Laplace's Equation

The Heat Equation

The heat or diffusion equation is an example of a *parabolic* partial differential equation,

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial x^2}, \qquad (*)$$

where κ is a positive constant known as the (thermal) diffusivity.
The heat equation has many applications, one of the most important being that it describes the evolution of temperature $\theta(x,t)$ in a one-dimensional conducting rod surrounded by a perfect insulator. Alternatively, θ could be the concentration of a dye in a channel of water, with $(*)$ describing its evolution in time.
Here we will concentrate on finding solutions of $(*)$ for $t > 0$ on a finite domain (e.g. $0 < x < L$) given specific *initial conditions* at $t = 0$

$$\theta(x, 0) = F(x).$$

Note that because the heat equation is first order in time, only one initial condition is needed. By contrast, the wave equation, which is second order in time, requires two sets of initial conditions for $z(x, 0)$ and $\partial z/\partial t(x, 0)$.

Boundary conditions

For unique solutions on the domain $0 < x < L$ we require boundary conditions at $x = 0$ and $x = L$. Two types of boundary conditions are common, depending on the physical situation at the boundary.

1. **Constant Temperature Boundary**: This boundary condition applies if a substance with a high heat capacity (e.g. water), and constant temperature T_0 is applied to one end of the rod. For example at $x = 0$
$$\theta(0, t) = T_0.$$

2. **Insulating Boundary**: This boundary condition applies if an insulator (e.g. wood, plastic) is applied to one end of the rod. There is then no heat flow through the boundary. Heat flow in the heat equation is proportional to $\partial\theta/\partial x$, so for example at $x = L$ an insulating boundary gives the boundary condition
$$\frac{\partial \theta}{\partial x}(L, t) = 0.$$

Steady Solutions

Steady (time-independent) solutions of $(*)$ under different boundary conditions can be found. For a steady solution $\theta = \theta_s(x)$ is a function of x only, and $(*)$ becomes

$$\frac{d^2\theta_s}{dx^2} = 0, \qquad \text{so} \qquad \theta_s(x) = Ax + B.$$

- For constant temperature boundary conditions $\theta_s(0) = T_1$ and $\theta_s(L) = T_2$ A and B can be determined and
$$\theta_s(x) = T_1 + (T_2 - T_1)\,\frac{x}{L}.$$
- If one boundary condition is insulating $d\theta_s/dx = 0$ at that boundary and $A = 0$. The steady solution must then be constant, and will be determined by the other boundary condition.

Steady solutions are important, because, as will be seen, all solutions of the heat equation evolve towards a particular steady solution.

Time-Dependent Solutions

Time-dependent solutions of $(*)$ can be found using the method of separation of variables. Writing $\theta(x,t) = X(x)T(t)$ and inserting in $(*)$ gives

$$\frac{dT}{dt}X = \kappa T\frac{d^2X}{dx^2}, \qquad \text{or} \qquad \frac{1}{\kappa T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda \quad \text{(cons)},$$

as we have equated a function of t only with a function of x only. It is easily shown that $\lambda > 0$ leads to solutions that grow exponentially in time. These are unphysical and may be ignored. The possibility $\lambda = 0$ recovers the steady solutions discussed above. Physical unsteady solutions therefore correspond to $\lambda = -p^2 < 0$, giving

$$\frac{d^2X}{dx^2} + p^2X = 0, \qquad \frac{dT}{dt} + p^2\kappa T = 0,$$

and

$$X(x) = A\cos px + B\sin px, \qquad T(t) = Ce^{-p^2\kappa t}$$

Unsteady solutions decay exponentially, and so solutions will converge to the appropriate steady solutions at long times $t \to \infty$.

Evolution from one steady state to another (Example 1)

As an example of a time-dependent problem, we consider the situation where the temperature in a rod evolves from one steady state to another.
Consider a rod of length L with one end held at temperature T_1 and the other at T_2 for $t < 0$. At $t = 0$ the end temperatures are changed suddenly to T_3 and T_4 respectively and remain so for $t > 0$. How does the temperature in the rod evolve in time?
At $t = 0$, the temperature is given by the steady state up to that point, so the initial conditions for the problem are

$$\theta(x, 0) = T_1 + (T_2 - T_1)\frac{x}{L}.$$

For $t > 0$ we expect the temperature to evolve towards the steady state for the new boundary conditions, so that as $t \to \infty$,

$$\theta \to \theta_s(x) = T_3 + (T_4 - T_3)\frac{x}{L}.$$

We therefore look for a solution

$$\theta(x, t) = \theta_s(x) + \theta_u(x, t),$$

where $\theta_u(x, t)$ denotes the unsteady, decaying, part of the solution for $t \geq 0$. Writing $\theta_u(x, t) = X(x)T(t)$ above, the boundary conditions at $x = 0, L$ imply that $X(0) = X(L) = 0$, giving $A = 0$ and $p = n\pi/L$ for $n = 1, 2, 3, \ldots$. Hence the general solution has the form

$$\theta(x, t) = \theta_s + \theta_u = T_3 + (T_4 - T_3)\frac{x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-n^2\pi^2\kappa t/L^2}.$$

The B_n can now be found using the initial conditions and Fourier series identities. At $t = 0$

$$\theta(x, 0) - \theta_s(x) = T_1 - T_3 + ((T_2 - T_4) - (T_1 - T_3))\frac{x}{L} = \theta_u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

Multiplying both sides by $\sin\{m\pi x/L\}$ and integrating in x from $x = 0$ to $x = L$, gives

$$\frac{LB_m}{2} = \int_0^L \left[T_1 - T_3 + ((T_2 - T_4) - (T_1 - T_3))\frac{x}{L}\right] \sin\frac{m\pi x}{L}\, dx$$

which can be integrated to give

$$B_m = \frac{2}{m\pi}\left((-1)^m(T_4 - T_2) - (T_3 - T_1)\right),$$

and the full time-dependent solution is

$$\theta(x, t) = \theta_s + \theta_u = T_3 + (T_4 - T_3)\frac{x}{L} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left((-1)^n(T_4 - T_2) - (T_3 - T_1)\right) \sin\frac{n\pi x}{L}\, e^{-n^2\pi^2\kappa t/L^2}.$$

Example 2: The ends of a rod of length L, and thermal diffusivity κ, are held at temperature $\theta = 0$ up to a given time ($t = 0$) after which they are held at temperature $\theta = T$. Find $\theta(x, t)$. (Note that this is a special case of Example 1).

$$\left[\textbf{Answer}: \quad \theta(x, t) = T - \frac{4T}{\pi}\sum_{m=0}^{\infty} \frac{1}{2m+1} \sin\left(\frac{(2m+1)\pi x}{L}\right) \exp\left(-\frac{(2m+1)^2\pi^2\kappa t}{L^2}\right)\right]$$

Laplace's Equation in Two Dimensions

Laplace's equation is an example of an *elliptic* partial differential equation. In two dimensions, and in Cartesian coordinates, it is

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \qquad (\dagger).$$

Laplace's equation has many applications, including in fluid dynamics, electricity and magnetism and gravitational potential theory. In keeping with the discussion of heat flow above, the steady state temperature distribution $\phi(x, y)$ in a two-dimensional conducting sheet of material, surrounded by an insulator, will satisfy Laplace's equation.
The form of the solution of Laplace's equation depends strongly on the geometry of the domain and the boundary conditions imposed (a general characteristic of elliptic equations). We will consider two different domain types: rectangular domains and circular domains.

Boundary conditions

Boundary conditions for Laplace's equation can be categorised as follows

1. **Dirichlet boundary conditions**: A Dirichlet boundary condition involves specifying ϕ everywhere on the boundary.

2. **Neumann boundary conditions**: A Neumann boundary condition involves specifying $\partial\phi/\partial n$ everywhere on the boundary, where $\partial\phi/\partial n$ denotes the directional derivative of ϕ in the direction normal to the boundary.

3. **Mixed boundary conditions**: If a boundary condition involves both ϕ and $\partial\phi/\partial n$ it is said to be of mixed type.

Laplace's Equation: Rectangular Domain

First, we will obtain solutions of Laplace's equation on a rectangular domain ($0 < x < a$, $0 < y < b$), with Dirichlet boundary conditions, i.e. ϕ specified on all four sides of the rectangle. In particular, we will start by considering problem (I) in the diagram, where $\phi = h(x)$ on the edge with $y = b$ and $\phi = 0$ on the other three sides.
Following the method of separation of variables, write $\phi(x, y) = X(x)Y(y)$. Then inserting in Laplace's equation

$$\frac{d^2X}{dx^2}Y + \frac{d^2Y}{dy^2}X = 0, \quad \text{or} \quad \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda \quad \text{cons},$$

since we have equated a function of x only and a function of y only. Three possibilities result in three different possible solutions:

- Case i: $\lambda = p^2 > 0$.
  This gives
  $$\frac{d^2X}{dx^2} - p^2X = 0, \qquad \frac{d^2Y}{dy^2} + p^2Y = 0,$$
  $$X(x) = A\cosh px + B\sinh px, \qquad Y(y) = C\cos py + D\sin py,$$
- Case ii: $\lambda = -p^2 < 0$.
  This gives
  $$\frac{d^2X}{dx^2} + p^2X = 0, \qquad \frac{d^2Y}{dy^2} - p^2Y = 0,$$
  $$X(x) = A\cos px + B\sin px, \qquad Y(y) = C\cosh py + D\sinh py,$$
- Case iii: $\lambda = 0$.
  $$\frac{d^2X}{dx^2} = 0, \qquad \frac{d^2Y}{dy^2} = 0,$$
  $$X(x) = Ax + B, \qquad Y(y) = Cy + D.$$

Cases (i-iii) together give the general solution of Laplace's equation in variables separable form.
Which of these solutions can be used to construct the solution to problem (I) in the rectangular domain? To satisfy the boundary conditions on the left and right sides, $X(x)$ must have $X(0) = X(a) = 0$. For cases (i) and (iii), this would result in $A = B = 0$, hence these do not contribute to the solution. For case (ii) however, non-zero solutions can be constructed with $A = 0$, $p = n\pi/a$, $n = 1, 2, 3, \ldots$. Note also that boundary condition on $y = 0$ gives $Y(0) = 0$, hence $C = 0$. The general solution can therefore be written (setting $B = 1$ w.l.o.g.)

$$\phi(x, y) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}. \qquad (**)$$

The boundary condition on $y = b$ has not yet been used, and this can be used to obtain the unknown constants D_n.

$$\phi(x, b) = h(x) = \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi b}{a}\sin\frac{n\pi x}{a}.$$

The D_ns can be obtained using the standard Fourier series technique - multiply both sides by $\sin(m\pi x/a)$, and integrating in x from 0 to a. This gives

$$D_m \frac{a}{2} \sinh \frac{m\pi b}{a} = \int_0^a h(x) \sin \frac{m\pi x}{a}\, dx.$$

The integral can be evaluated for any given $h(x)$ and the solution for problem (I) is then given by $(**)$ above.

Symmetry and Linearity

The solutions to problems (II-IV) in the diagram can be obtained from the solution to problem (I) using symmetry arguments. For (II) the solution is identical to $(**)$ above except with the variable $(b - y)$ substituted for y, and $f(x)$ substituted for $h(x)$, i.e.

$$\phi(x,y) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}, \qquad D_n = \frac{2}{a \sinh \frac{n\pi b}{a}} = \int_0^a f(x) \sin \frac{n\pi x}{a}\, dx.$$

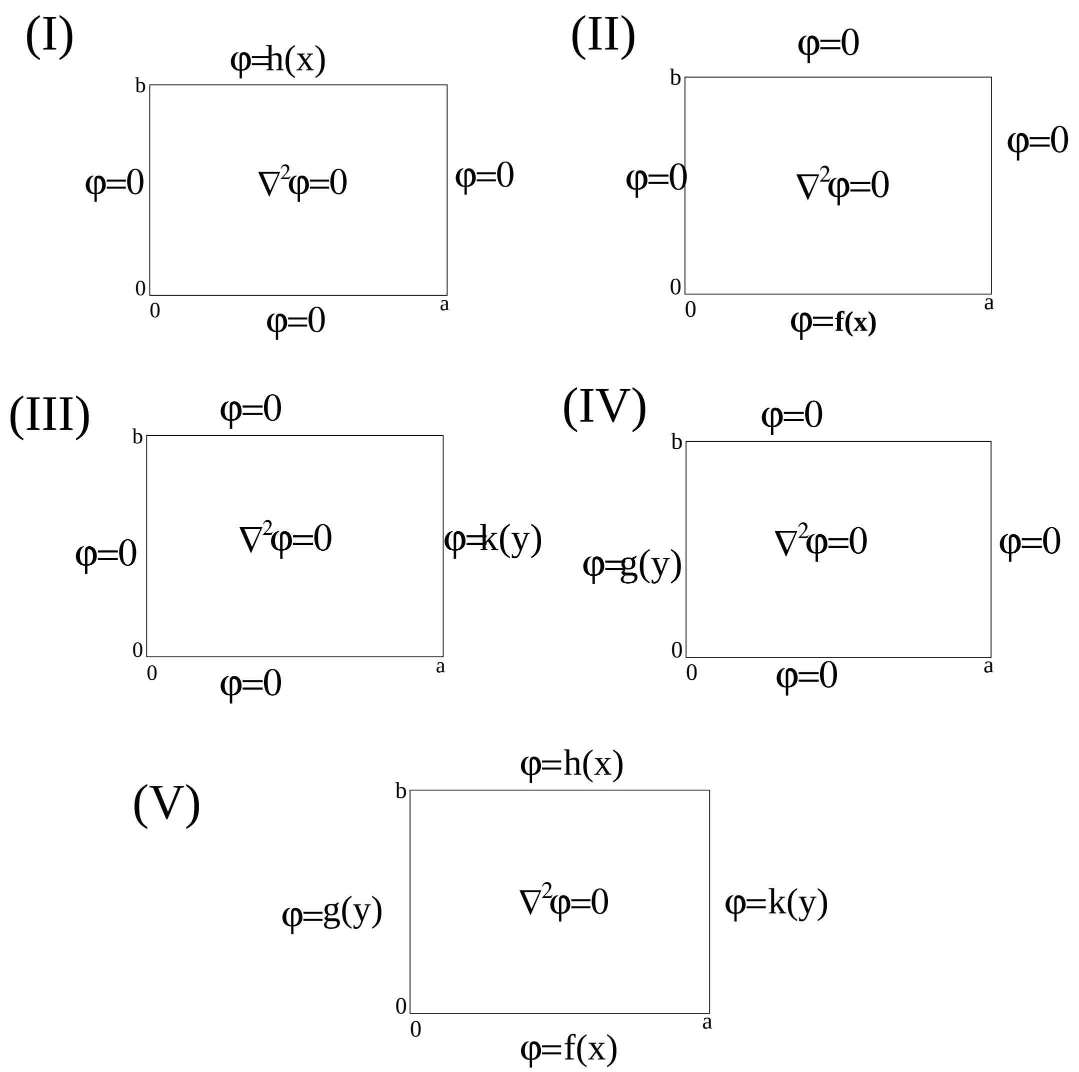
In effect the solution (I) has just been reflected about the axis of symmetry at $y = b/2$.
For problems (III) and (IV) we require case (i) solutions above in order to satisfy the top and bottom boundary conditions. However, because there is nothing special about our choice of x and y directions, symmetry implies that x and y (together with a and b) can just be switched in the solutions for (I) and (II), to give the solutions for (III) and (IV) respectively. For example (without needing to do any more work) the case (III) solution is

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b}, \qquad B_n = \frac{2}{b \sinh \frac{n\pi a}{b}} = \int_0^b g(y) \sin \frac{n\pi y}{b}\, dy.$$

To get the solution for the most general problem (V), we can use the fact that Laplace's equation is linear. Linearity means that

$$\nabla^2 (\phi_I + \phi_{II} + \phi_{III} + \phi_{IV}) = \nabla^2 \phi_I + \nabla^2 \phi_{II} + \nabla^2 \phi_{III} + \nabla^2 \phi_{IV} = 0,$$

where ϕ_I, ϕ_{II}, etc. denote the solutions to problems (I-IV) above. Hence $\phi_V = \phi_I + \phi_{II} + \phi_{III} + \phi_{IV}$ is also a solution of Laplace's equation. Clearly ϕ_V satisfies the desired boundary conditions on all four edges, as $\phi_V = \phi_I + 0 + 0 + 0 = h(x)$ on $y = b$, $\phi_V = 0 + \phi_{II} + 0 + 0 = f(x)$ on $y = 0$, etc. Hence ϕ_V is the solution to problem (V).



Laplace's equation on a rectangular domain with a series of representative Dirichlet boundary conditions. The solution ϕ of each problem gives the steady temperature distribution in a rectangular sheet of metal, surrounded by insulator, with edges held at the given temperatures.

Laplace's Equation: Circular Domain

In a circular domain it is convenient to work in plane polar coordinates (r, θ). In plane polar co-ordinates, Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0, \qquad (\ddagger).$$

Using the method of separation of variables, as above, we seek solutions of the form

$$\phi(r, \theta) = R(r)\Theta(\theta).$$

As we are on a circular domain, with $0 \leq \theta < 2\pi$, for continuity of ϕ we require $\Theta(\theta) = \Theta(\theta + 2\pi)$. (Note that this condition would not apply if, for example, the domain was a 'wedge' with $0 \leq \theta \leq \alpha$). Inserting into $(\ddagger)$,

$$\frac{\Theta}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0,$$

and dividing by $R\Theta/r^2$ and rearranging,

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = \lambda \qquad \text{(constant)},$$

as if we have equality between a function of r and a function of θ then they must each equal a constant (λ).
Again, three possibilities for λ result in three different possible solutions:

- Case i: $\lambda = p^2 > 0$.
  This gives
  $$\frac{d^2\Theta}{d\theta^2} - p^2\Theta = 0,$$
  $$\Theta(\theta) = A\cosh p\theta + B\sinh p\theta,$$
  It is clear there are no choices for A, B satisfying $\Theta(\theta) = \Theta(\theta + 2\pi)$, so case i solutions can be discounted.

- Case ii: $\lambda = -p^2 < 0$.
  This gives
  $$\frac{d^2\Theta}{d\theta^2} + p^2\Theta = 0, \qquad r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - p^2R = 0,$$
  $$\Theta(\theta) = A\cos p\theta + B\sin p\theta, \qquad R(r) = Cr^p + Dr^{-p},$$
  In this case, the solutions will satisfy the periodicity condition $\Theta(\theta) = \Theta(\theta + 2\pi)$ if $p = n$, for integer $n = 1, 2, 3, 4, \ldots$.

- Case iii: $\lambda = 0$.
  $$\frac{d^2\Theta}{d\theta^2} = 0, \qquad \frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0,$$
  $$\Theta(\theta) = A\theta + B, \qquad R(r) = C + D\log r.$$
  The periodicity condition $\Theta(\theta) = \Theta(\theta + 2\pi)$ implies that we must have $A = 0$.

Combining the case ii and iii solutions gives the general solution for Laplace's equation on a circular domain,

$$\phi(r,\theta) = C_0 + D_0 \log r + \sum_{n=1}^{\infty} \left\{ C_n r^n + \frac{D_n}{r^n} \right\} \{A_n \cos n\theta + B_n \sin n\theta\}.$$

Boundary conditions can be used to obtain the unknown constants A_n, B_n, C_n, D_n (note that there is some redundancy in the use of four constants for each value of n, only three are really needed!).

Example 3: The temperature ϕ of a circular sheet of metal of unit radius satisfies Laplace's equation in $r < 1$. On the edge of the sheet ($r = 1$) the temperature is held at $\phi = 2\cos\theta$. Find the temperature everywhere in the sheet. (Note that the temperature at the centre of the sheet must be finite).

$$[\textbf{Answer}: \quad \phi(r,\theta) = 2r\cos\theta.]$$

Example 4: A function $\phi(r,\theta)$ satisfies Laplace's equation both inside and outside the unit circle. Inside the unit circle $\phi = \phi_1$ satisfies $r\phi_1 \to \cos\theta$ as $r \to 0$. Outside the unit circle $\phi = \phi_2$ satisfies $\phi_2 - r\cos\theta \to 2$ as $r \to \infty$. On the unit circle itself ($r = 1$), $\phi_1 = \phi_2$ and

$$\frac{\partial \phi_1}{\partial r} - \frac{\partial \phi_2}{\partial r} = 1.$$

Find $\phi_1(r,\theta)$ and $\phi_2(r,\theta)$.

$$\left[\textbf{Answer}: \quad \phi_1(r,\theta) = 2 + \log r + \left(r + \frac{1}{r}\right)\cos\theta \quad \text{in} \quad 0 < r \le 1.\right]$$

$$\left[\qquad \phi_2(r,\theta) = 2 + \left(r + \frac{1}{r}\right)\cos\theta \quad \text{in} \quad r \ge 1.\right]$$